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Manuel Núñez

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Uniform estimates on the velocity in Rayleigh–Bénard convection

Manuel Núñez[a)]
Departamento de Análisis Matemático, Universidad de Valladolid, 47005 Valladolid, Spain

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The kinetic energy of a fluid located between two plates at different temperatures is easily bounded by classical inequalities. However, experiments and numerical simulations indicate that when the convection is turbulent, the volume of the domains in which the speed is large, is rather small. This could imply that the maximum of the speed, in contrast with its quadratic mean, does not admit an *a priori* upper bound. It is proved that, provided the pressure remains bounded, a uniform estimate for the speed maximum does indeed exist, and that it depends on the maxima of certain ratios between temperature, pressure, and velocity. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1855400]

I. INTRODUCTION

The study of thermal convection of a fluid powered by the difference of temperature between two plates, known as Rayleigh–Bénard convection, has been an extensively studied subject for a long time. Computer modeling and physical experiments have produced an enormous wealth of information: for recent reviews, see Refs. 1 and 2. Perhaps unavoidably, there has not been a comparable volume of rigorous studies, if we except the study of the stability of different patterns (Ref. 3, pp. 23–95). It is well known that when the difference of temperature between the top and bottom plates exceeds a certain amount, usually measured in terms of the Rayleigh constant R, convection sets in. Near the onset, convection cells occur; with increasing R, and depending also on the ratio between viscosity and thermal diffusivity (the Prandtl number) more complex patterns appear and bifurcations to chaotic states may occur. The same may be said of the temperature: for a colorful illustration, starting with regular rolls, see e.g., Ref. 4.

The standard mathematical model of Reyleigh–Bénard convection is given by the Boussinesq approximation to the equations of motion, which we repeat here for convenience. We will consider a d-dimensional domain U of the form $\Omega\times[0,h]$, and as usual we will assume that the temperature is constant at the lower and upper lids, $T=T_0$ at $\Omega\times\{0\}$ and $T=T_h<T_0$ at $\Omega\times\{h\}$. The rest of the boundary conditions will be discussed later. Let us denote by $\mathbf{v}$ the fluid velocity, T the temperature, ν the kinematic viscosity, κ the thermal diffusivity, and π the kinetic pressure. Then the nondimensionalized Boussinesq approximation (see, e.g., Ref. 5) to the equations of motion is

$$\frac{\partial \mathbf{v}}{\partial t} = \nu\Delta\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - \nabla\pi + (T-T_h)\mathbf{e}_d, \tag{1}$$

$$\frac{\partial T}{\partial t} = \kappa\Delta T - \mathbf{v}\cdot\nabla T, \tag{2}$$

[a)]Electronic mail: mnjmhd@am.uva.es

$$\nabla \cdot \mathbf{v} = 0. \tag{3}$$

$\mathbf{e}_d$ denotes the vertical unit vector. Traditionally the difference θ of the actual temperature with the linear one between the lids (associated to pure heat conduction) is used,

$$\begin{aligned} \theta &= T - T_0 + \beta x_d, \\ \beta &= \frac{T_0 - T_h}{h}. \end{aligned} \tag{4}$$

π is also changed to $\pi+\beta x_d^2/2-(T_0-T_h)x_d$ (which we denote again by π). The final system is

$$\frac{\partial \mathbf{v}}{\partial t} = \nu\Delta\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} - \nabla\pi + \theta\mathbf{e}_d, \tag{5}$$

$$\frac{\partial \theta}{\partial t} = \kappa\Delta\theta - \mathbf{v}\cdot\nabla\theta + \beta v_d, \tag{6}$$

$$\nabla \cdot \mathbf{v} = 0. \tag{7}$$

Boundary conditions are usually the following ones: the upper and lower plates are taken either rigid, where we assume a no-slip condition and $\mathbf{v}$ vanishes there, or stress free, in which case $v_d=0$ and the vertical derivatives of the remaining components of the velocity are also zero, $\partial_d v_i=0$. The lateral walls are assumed rigid and conducting, so that $\mathbf{v}$ and θ vanish there.

Let us state some classical results, since T satisfies (2), which is a scalar parabolic equation without terms in T, it also satisfies the maximum and minimum principles (see, e.g., Ref. 6). That means that T lies always between T_0 and T_h, which makes excellent physical sense. Therefore θ is uniformly bounded. By multiplying (5) by $\mathbf{v}$, integrating in U and making use of the boundary conditions, one gets

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_U v^2\,\mathrm{d}V + \nu\int_U |\nabla\mathbf{v}|^2\,\mathrm{d}V = \int_U \theta v_d\,\mathrm{d}V \leqslant \|\theta\|_\infty\,\mathrm{Vol}(U)^2\|v_d\|_2, \tag{8}$$

where $\mathrm{Vol}(U)$ denotes the volume (area in dimension two) of U. Since with our boundary conditions a Poincaré inequality holds, there exists a positive constant c such that

$$c\int_U |v^2|\mathrm{d}V \leqslant \int_U |\nabla\mathbf{v}|^2\,\mathrm{d}V.$$

Thus, by standard inequalities,

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\mathbf{v}\|_2^2 + \frac{\nu c}{2}\|\mathbf{v}\|_2^2 \leqslant \frac{1}{2\nu^2c^2}\|\theta\|_\infty^2\,\mathrm{Vol}(U), \tag{9}$$

which implies that $\|\mathbf{v}\|_2$ is bounded for all time.

As in many other turbulent situations, modeling of the chaotic phase of convection shows a tendency of the flow to concentrate the velocity in regions of small volume.[7] Thus the boundedness of the kinetic energy does not provide an *a priori* bound upon the maximum of the speed. It is true that physically it seems obvious that this maximum cannot grow without limit, but nevertheless it is interesting to obtain rigorous estimates in terms of the main magnitudes of the problem. Our only hypothesis will be the boundedness of the pressure π [or, to be specific, of $\pi/(1+v)$].

II. ANALYSIS OF THE MOMENTS OF THE VELOCITY

Let us start with the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \Delta \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{v} - \nabla \pi + \theta \mathbf{e}_d, \tag{10}$$

and for $p=1,2\ldots$, let

$$F_p = \int_U v^p \, \mathrm{d}V, \tag{11}$$

where $v=|\mathbf{v}|$ represents the modulus of $\mathbf{v}$. F_p is a function of time. Since $v^2=\mathbf{v}\cdot\mathbf{v}$,

$$\frac{\partial v^{2p}}{\partial t} = 2p v^{2p-2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t}.$$

Therefore

$$\frac{1}{2p}\dot{F}_{2p} = \int_U v^{2p-2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \mathrm{d}V,$$

and taking into account the momentum equation,

$$\frac{1}{2p}\dot{F}_{2p} = -\int_U v^{2p-2}(\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \mathbf{v} \, \mathrm{d}V + \nu \int_U v^{2p-2} \mathbf{v} \cdot \Delta \mathbf{v} \, \mathrm{d}V + \int_U (\theta v^{2p-2} v_d - v^{2p-2} \mathbf{v} \cdot \nabla \pi) \mathrm{d}V. \tag{12}$$

In the first place,

$$\int_U v^{2p-2}(\mathbf{v} \cdot \nabla \mathbf{v}) \cdot \mathbf{v} \, \mathrm{d}V = \int_U \frac{1}{2p} \mathbf{v} \cdot \nabla v^{2p} \, \mathrm{d}V = 0. \tag{13}$$

As for the dissipative term,

$$\begin{aligned}\int_U v^{2p-2} \mathbf{v} \cdot \Delta \mathbf{v} \, \mathrm{d}V &= \int_U \sum_j (\nabla \cdot (v_j v^{2p-2} \nabla v_j) - \nabla v_j \cdot \nabla (v^{2p-2} v_j)) \mathrm{d}V \\ &= \frac{1}{2}\int_{\partial U} v^{2p-2} \frac{\partial v^2}{\partial n} \mathrm{d}\sigma - \int_U \left(v^{2p-2} |\nabla \mathbf{v}|^2 + \frac{p-1}{2} v^{2p-4} |\nabla v^2|^2 \right) \mathrm{d}V. \end{aligned} \tag{14}$$

It is understood that the last term (multiplied by $p-1$) vanishes when $p=1$; there are never negative powers of v. As for the boundary integral, it also vanishes, since $\partial v^2/\partial n$ vanishes in all the boundary, including possible stress-free surfaces.

The last term we must consider is

$$\int_U (\theta v^{2p-2} v_d - v^{2p-2} \mathbf{v} \cdot \nabla \pi) \mathrm{d}V. \tag{15}$$

Since

$$-\int_U v^{2p-2} \mathbf{v} \cdot \nabla \pi \, \mathrm{d}V = -\int_{\partial U} v^{2p-2} \pi \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\sigma + \int_U (p-1) \pi v^{2p-4} \mathbf{v} \cdot \nabla v^2 \, \mathrm{d}V, \tag{16}$$

and again the boundary integral vanishes, the term in (16) is

$$\int_U (v^{2p-2} \theta v_d + (p-1) \pi v^{2p-4} \mathbf{v} \cdot \nabla v^2) \mathrm{d}V, \tag{17}$$

with the same meaning as before when $p=1$. We may bound (17) in several ways. We first choose

$$\begin{aligned}
\left| \int_U v^{2p-2} \theta v_d + (p-1) \pi v^{2p-4} \mathbf{v} \cdot \nabla v^2 \, \mathrm{d}V \right| &= \left| \int_U \frac{\theta}{1+v} (v_d v^{2p-2} + v_d v^{2p-1}) + (p-1) \frac{\pi}{1+v} (v^{2p-4} \right. \\
&\quad \left. + v^{2p-3}) \mathbf{v} \cdot \nabla v^2 \, \mathrm{d}V \right| \\
&\leqslant \left\| \frac{\theta}{1+v} \right\|_\infty \int_U (v^{2p-2} + v^{2p-1}) |v_d| \mathrm{d}V + (p-1) \left\| \frac{\pi}{1+v} \right\|_\infty \\
&\quad \times \int_U (v^{2p-4} + v^{2p-3}) |\mathbf{v} \cdot \nabla v^2| \mathrm{d}V \\
&\leqslant \left\| \frac{\theta}{1+v} \right\|_\infty \int_U (v^{2p-1} + v^{2p}) \mathrm{d}V + (p-1) \left\| \frac{\pi}{1+v} \right\|_\infty \\
&\quad \times \int_U (v^{2p-3} + v^{2p-2}) |\nabla v^2| \mathrm{d}V.
\end{aligned} \tag{18}$$

From now on we will denote

$$\alpha = \left\| \frac{\theta}{1+v} \right\|_\infty,$$

$$\beta = \left\| \frac{\pi}{1+v} \right\|_\infty.$$

Notice that they are functions of t. Thus

$$\begin{aligned}
\frac{1}{2p} \dot{F}_{2p} &\leqslant -\nu \int_U v^{2p-2} |\nabla \mathbf{v}|^2 \, \mathrm{d}V - \frac{\nu(p-1)}{2} \int_U v^{2p-4} |\nabla v^2|^2 \, \mathrm{d}V + \alpha \int_U (v^{2p-1} + v^{2p}) \mathrm{d}V \\
&\quad + (p-1) \beta \int_U (v^{2p-3} + v^{2p-2}) |\nabla v^2| \mathrm{d}V.
\end{aligned} \tag{19}$$

For our first estimate we will not make use of the first dissipative term. We have

$$\alpha \int_U (v^{2p-2} + v^{2p-1}) \mathrm{d}V \leqslant \alpha (F_{2p-1} + F_{2p}). \tag{20}$$

As for the second term, by using Cauchy–Schwarz and Young's inequalities,

$$
\begin{aligned}
(p-1)\beta\int_U (v^{p-1}+v^p)v^{p-2}|\nabla v^2|\mathrm{d}V &\leqslant (p-1)\beta\left(\int_U (v^{p-1}+v^p)^2\,\mathrm{d}V\right)^{1/2}\left(\int_U v^{2p-4}|\nabla v^2|^2\,\mathrm{d}V\right)^{1/2}\\
&\leqslant (p-1)\beta^2\frac{1}{\nu}\int_U (v^{p-1}+v^p)^2\,\mathrm{d}V\\
&\quad+\frac{(p-1)\nu}{4}\int_U v^{2p-4}|\nabla v^2|^2\,\mathrm{d}V\\
&\leqslant (p-1)\beta^2\frac{2}{\nu}(F_{2p-2}+F_{2p})+\frac{(p-1)\nu}{4}\int_U v^{2p-4}|\nabla v^2|^2\,\mathrm{d}V.
\end{aligned}
\tag{21}
$$

We have proved the recursive inequality

$$
\frac{1}{2p}\dot{F}_{2p}\leqslant-\frac{\nu(p-1)}{4}\int_U v^{2p-4}|\nabla v^2|^2\,\mathrm{d}V+\alpha F_{2p-1}+\alpha F_{2p}+\frac{2(p-1)\beta^2}{\nu}F_{2p-2}+\frac{2(p-1)\beta^2}{\nu}F_{2p}.
\tag{22}
$$

We use now the fact that U has finite volume to bound all the F_k in terms of F_{2p},

$$
\begin{aligned}
F_{2p-2}&\leqslant \mathrm{Vol}(U)^{1/p}F_{2p}^{1-1/p},\\
F_{2p-1}&\leqslant \mathrm{Vol}(U)^{1/2p}F_{2p}^{1-1/2p}.
\end{aligned}
\tag{23}
$$

Let us begin studying a series of alternatives. It may happen

(A) $F_{2p}<1$. Otherwise,
(B) $F_{2p-2}\leqslant \mathrm{Vol}(U)^{1/p}F_{2p}$, $F_{2p-1}\leqslant \mathrm{Vol}(U)^{1/2p}F_{2p}$.

We will consider the consequences of alternative (B). We have

$$
\frac{1}{2p}\dot{F}_{2p}+\frac{\nu(p-1)}{4}\int_U v^{2p-4}|\nabla v^2|^2\,\mathrm{d}V\leqslant\left(\alpha(1+m(U)^{1/2p})+\frac{2(p-1)\beta^2}{\nu}(1+\mathrm{Vol}(U)^{1/p})\right)F_{2p},
\tag{24}
$$

and if we call $k=1+\max\{\mathrm{Vol}(U),1\}$,

$$
\frac{1}{2p}\dot{F}_{2p}+\frac{\nu(p-1)}{4}\int_U v^{2p-4}|\nabla v^2|^2\,\mathrm{d}V\leqslant k\left(\alpha+\frac{2(p-1)\beta^2}{\nu}\right)F_{2p}.
\tag{25}
$$

Let us now remember a particular case of the Gagliardo–Nirenberg inequality (Ref. 8, pp. 69 and 70). For any function $f\in H^1(U)$, there exists a constant C depending only on U such that

$$
\|f\|_2^{d+2}\leqslant C(\|\nabla f\|_2+\|f\|_1)^d\|f\|_1^2.
\tag{26}
$$

Since $(x+y)^d\leqslant 2^{d-1}(x^d+y^d)$, by taking $\lambda=\max\{2^{d-1}C,1\}$,

$$
\|f\|_2^{d+2}\leqslant \lambda(\|\nabla f\|_2^d+\|f\|_1^d)\|f\|_1^2,
\tag{27}
$$

with $\lambda\geqslant 1$. Notice that for $f=v^p$,

$$
\|f\|_2=F_{2p}^{1/2},\quad \|f\|_1=F_p,\quad \nabla f=\frac{p}{2}v^{2p-2}\nabla v^2,\quad \|\nabla f\|_2^2=\frac{p^2}{4}\int_U v^{2p-4}|\nabla v^2|^2\,\mathrm{d}V.
\tag{28}
$$

In the inequality (27), there exist two alternatives. Either

$$\text{(b1)}\quad \|f\|_2 < \lambda^{1/(d+2)}\|f\|_1, \tag{29}$$

or

$$\text{(b2)}\quad \|\nabla f\|_2^2 \geqslant \left(\frac{\|f\|_2^{d+2} - \lambda\|f\|_1^{d+2}}{\lambda\|f\|_1^2}\right)^{2/d}. \tag{30}$$

With our election of f, the first alternative means

$$F_{2p} < \lambda^{2/(d+2)}F_p^2. \tag{31}$$

Alternative (b2), when taken into (25), yields

$$\frac{1}{2p}\dot{F}_{2p} \leqslant -\frac{\nu(p-1)}{p^2}\left(\frac{F_{2p}^{(d+2)/2} - \lambda F_p^{d+2}}{\lambda F_p^2}\right)^{2/d} + k\left(\alpha + \frac{2(p-1)\beta^2}{\nu}\right)F_{2p}. \tag{32}$$

Take now, for $p \geqslant 2$,

$$\gamma_p = \lambda\left(1 + k^{d/2}\left(\frac{p^2}{\nu(p-1)}\right)^{d/2}\left(\alpha + \frac{2(p-1)\beta^2}{\nu}\right)^{d/2}\right). \tag{33}$$

Notice that $\gamma_p \geqslant \lambda \geqslant \lambda^{2/(d+2)} \geqslant 1$. A short calculation will convince us that if $F_{2p} > \gamma_p F_p^2$ and (b2) occurs, then $\dot{F}_{2p} \leqslant 0$. Since alternative (b1) is included in

$$\text{(B1)}\quad F_{2p} \leqslant \gamma_p F_p^2,$$

the remaining possibility is

$$\text{(B2)}\quad \dot{F}_{2p} \leqslant 0.$$

Recall that all this assumes (B). Therefore the alternatives are (A) $F_{2p} \leqslant 1$, (B1) or (B2).

III. UNIFORM ESTIMATES IN TIME

Let us consider a time interval $[0,\tau]$, and let

$$\begin{aligned} \Phi_p(\tau) &= \max\{1, \max\{F_p(t) : t \in [0,\tau]\}\} \\ \Gamma_p(\tau) &= \max\{\gamma_p(t) : t \in [0,\tau]\}. \end{aligned} \tag{34}$$

For every $t \in [0,\tau]$ where (A) or (B1) occurs, certainly

$$F_{2p}(t) \leqslant \Gamma_p(\tau)\Phi_p(\tau)^2. \tag{35}$$

Now, if (35) occurs for every $t \in [0,\tau]$, obviously

$$\Phi_{2p}(\tau) \leqslant \Gamma_p(\tau)\Phi_p(\tau)^2. \tag{36}$$

The other possibility is that for a certain $t_1 \in [0,\tau]$,

$$F_{2p}(t_1) > \Gamma_p(\tau)\Phi_p(\tau)^2, \tag{37}$$

which implies that (B2) holds, i.e., $\dot{F}_{2p}(t_1) \leqslant 0$. Let $(t_0, t_1]$ be a maximal left interval where (37) occurs. We know that F_{2p} is decreasing there. If $t_0 > 0$, $F_{2p}(t_0) = \Gamma_p(\tau)\Phi_p(\tau)^2$, which, since $F_{2p}(t_1) \leqslant F_{2p}(t_0)$, contradicts our hypothesis. The only possibility is that (35) occurs nowhere, i.e., $t_0 = 0$. In that case, F_{2p} is decreasing in $[0, t_1]$ and therefore $F_{2p}(t_1) \leqslant F_{2p}(0)$. Thus, in every case

$$\Phi_{2p}(\tau) \leqslant \max\{\Gamma_p(\tau)\Phi_p(\tau)^2, F_{2p}(0)\}. \tag{38}$$

Since $\|\mathbf{v}(t)\|_p = F_p(t)^{1/p}$, if we denote

$$\phi_p(\tau) = \max\{1, \max\{\|\mathbf{v}(t)\|_p : t \in [0,\tau]\}\}, \tag{39}$$

the $2p$th root of (38) yields

$$\phi_{2p}(\tau) \leqslant \max\{\Gamma_p(\tau)^{1/2p}\phi_p(\tau), \|\mathbf{v}(0)\|_{2p}\}. \tag{40}$$

Hence

$$\begin{gathered}
\phi_4(\tau) \leqslant \max\{\Gamma_2(\tau)^{1/4}\phi_2(\tau),\ \|\mathbf{v}(0)\|_4\}, \\
\phi_8(\tau) \leqslant \max\{\Gamma_4(\tau)^{1/8}\Gamma_2(\tau)^{1/4}\phi_2(\tau), \Gamma_4(\tau)^{1/8}\|\mathbf{v}(0)\|_4,\ \|\mathbf{v}(0)\|_8\}\cdots, \\
\phi_{2^n}(\tau) \leqslant \{\Gamma_{2^{n-1}}(\tau)^{1/2^n}\Gamma_{2^{n-2}}(\tau)^{1/2^{n-1}}\cdots\Gamma_2(\tau)^{1/4}\phi_2(\tau), \\
\Gamma_{2^{n-1}}(\tau)^{1/2^n}\Gamma_{2^{n-2}}(\tau)^{1/2^{n-1}}\cdots\Gamma_4(\tau)^{1/8}\|\mathbf{v}(0)\|_4\ldots,\ \|\mathbf{v}(0)\|_{2^n}.
\end{gathered} \tag{41}$$

Let us study the infinite product

$$\Gamma_2(\tau)^{1/4}\Gamma_4(\tau)^{1/8}\cdots\Gamma_{2^{n-1}}(\tau)^{1/2^n}\cdots. \tag{42}$$

Recall that

$$\alpha(t) = \left\|\frac{\theta(t)}{1+v(t)}\right\|_\infty, \quad \beta(t) = \left\|\frac{\pi(t)}{1+v(t)}\right\|_\infty. \tag{43}$$

Let us denote by $\overline{\alpha}(\tau)$, $\overline{\beta}(\tau)$ their respective maxima in $[0,\tau]$. By the expression in (33),

$$\begin{aligned}
\Gamma_{2^n}(\tau) &\leqslant \lambda\left(1 + k^{d/2}\left(\frac{2^{2n}}{\nu(2^n-1)}\right)^{d/2}\left(\overline{\alpha}(\tau) + \frac{2(2^n-1)}{\nu}\overline{\beta}(\tau)^2\right)^{d/2}\right) \\
&\leqslant \lambda\left(1 + \left(2^{2n+2}k\left(\frac{\overline{\alpha}(\tau)}{\nu} + \frac{\overline{\beta}(\tau)^2}{\nu^2}\right)\right)^{d/2}\right) \leqslant \lambda\left(1 + \left(2^{2n+2}k\left(\frac{\overline{\alpha}(\tau)}{\nu} + \frac{\overline{\beta}(\tau)^2}{\nu^2} + 1\right)\right)^{d/2}\right).
\end{aligned} \tag{44}$$

The addition of 1 to the parentheses is intended to ensure that

$$x = \left(2^{2n+2}k\left(\frac{\overline{\alpha}(\tau)}{\nu} + \frac{\overline{\beta}(\tau)^2}{\nu^2} + 1\right)\right)^{d/2} \geqslant 1.$$

Since for those x, $\log(1+x) \leqslant 1 + \log x$ holds,

$$\log \Gamma_{2^n}(\tau) \leqslant \log\lambda + 1 + \frac{d}{2}\left(\log k + (2n+2)\log 2 + \log\left(\frac{\overline{\alpha}(\tau)}{\nu} + \frac{\overline{\beta}(\tau)^2}{\nu^2} + 1\right)\right).$$

Hence the sum of the logarithms of Γ_{2^n} satisfies

$$\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\log\Gamma_{2^n}(\tau) \leqslant \frac{1}{2}\left(1 + \log\lambda + \frac{d}{2}\log k\right) + \frac{d}{2}\log 2\sum_{n=1}^{\infty}\frac{2n+2}{2^{n+1}} + \frac{d}{4}\log\left(\frac{\overline{\alpha}(\tau)}{\nu} + \frac{\overline{\beta}(\tau)^2}{\nu^2} + 1\right). \tag{45}$$

Thus, since the sum of the series $\Sigma(2n+2)/2^{n+1}$ is 3,

$$\prod_{n=1}^{\infty}\Gamma_{2^n}(\tau)^{1/2^{n+1}} \leqslant 2^{3d/2}(\lambda e)^{1/2}k^{d/4}\left(\frac{\overline{\alpha}(\tau)}{\nu} + \frac{\overline{\beta}(\tau)^2}{\nu^2} + 1\right)^{d/4}, \tag{46}$$

and the same may be said of any finite product

$$\prod_{n=1}^{m} \Gamma_{2^n}(\tau)^{1/2^{n+1}}. \tag{47}$$

Since, on the other hand,

$$\|\mathbf{v}(0)\|_p \leqslant \|\mathbf{v}(0)\|_\infty \,\mathrm{Vol}(U)^{1/p} \leqslant k\|\mathbf{v}(0)\|_\infty, \tag{48}$$

we find

$$\phi_{2^n}(\tau) \leqslant 2^{3d/2}(\lambda e)^{1/2}k^{d/4}\left(\frac{\overline{\alpha}(\tau)}{\nu} + \frac{\overline{\beta}(\tau)^2}{\nu^2} + 1\right)^{d/4} \max\{\phi_2(\tau), k\|\mathbf{v}(0)\|_\infty\}. \tag{49}$$

It is well known that $\|\mathbf{v}\|_p \to \|\mathbf{v}\|_\infty$ as $p\to\infty$, so that $\phi_{2^n}(\tau) \to \max\{1, \|\mathbf{v}(t)\|_\infty : t\in[0,\tau]\}$. Also, $\phi_2(\tau)$ represents the maximum of the kinetic energy in $[0,\tau]$, which we denote by $E(\tau)$. Calling M the universal constant written in (49), we obtain our main estimate

$$\begin{aligned}\max\{\|\mathbf{v}(t)\|_\infty : t\in[0,\tau]\} \leqslant M\left(\frac{1}{\nu}\max_{[0,\tau]}\left\|\frac{\theta(t)}{1+v(t)}\right\|_\infty + \frac{1}{\nu^2}\max_{[0,\tau]}\left\|\frac{\pi(t)}{1+v(t)}\right\|_\infty^2 + 1\right)^{d/4} \\ \times \max\{E(\tau), k\|\mathbf{v}(0)\|_\infty\}.\end{aligned} \tag{50}$$

IV. COMMENTS AND EXTENSIONS OF THE ESTIMATES

In principle it could look as if the estimates in terms of θ and π divided by $1+v$ are unnecessary, since if v is bounded so are θ and π. While there is no conceptual gain in taking these magnitudes divided by $1+v$, the estimate (50) is in fact finer than one involving only $\|\theta\|_\infty$ and $\|\pi\|_\infty$. It can be far better if the regions where θ and/or π are larger coincide with regions of high velocity. In particular, high temperature deviation propels the fluid faster, so it is likely that $\theta/(1+v)$ is considerably smaller than θ.

When the flow is chaotic, the temperature may become irregular. Therefore it is possible that some primitive of θ (e.g., a function Θ such that for some coordinate j, $\partial_j\Theta=\theta$) may have a better behavior than the temperature deviation: portions where θ is positive may compensate with others where it is negative to obtain a smooth result. We will see that $\|\mathbf{v}\|_\infty$ may also be bounded in terms of $\Theta/(1+v)$. The method follows the steps of the previous one: the term

$$\int_U \theta v^{2p-2} v_d \,\mathrm{d}V \tag{51}$$

may be written as

$$\int_U (\partial_j\Theta) v^{2p-2} v_d \,\mathrm{d}V = -\int_U \Theta\partial_j(v^{2p-2}v_d)\mathrm{d}V = -\int_U \Theta(v^{2p-2}\partial_j v_d + 2(p-1)v^{2p-4}v_d\mathbf{v}\cdot\partial_j\mathbf{v})\mathrm{d}V. \tag{52}$$

This may be bounded by

$$(2p-1)\int_U \left|\frac{\Theta}{1+v}\right|(v^{2p-2}+v^{2p-1})|\nabla \mathbf{v}|\mathrm{d}V \leqslant (2p-1)\left\|\frac{\Theta}{1+v}\right\|_\infty \int_U (v^{2p-2}+v^{2p-1})|\nabla\mathbf{v}|\mathrm{d}V. \tag{53}$$

Using now the Cauchy–Schwarz inequality, the term is bounded by

$$(2p-1)^2\left\|\frac{\Theta}{1+v}\right\|_\infty \frac{1}{2\nu}\int_U (v^{2p-2}+v^{2p})\mathrm{d}V+\nu\int_U v^{2p-2}|\nabla \mathbf{v}|^2\,\mathrm{d}V. \tag{54}$$

The last term may now be cancelled with the first of the dissipative terms (which we did not use in our previous proof) and we are left with

$$(2p-1)^2\left\|\frac{\Theta}{1+v}\right\|_\infty^2 \frac{1}{2\nu}(F_{2p-2}+F_{2p}). \tag{55}$$

The rest of the proof is analogous to the previous one. We are left with a bound of the form

$$\max\{\|\mathbf{v}(t)\|_\infty : t\in[0,\tau]\} \leqslant M\left(\frac{1}{\nu^2}\max_{[0,\tau]}\left\|\frac{\Theta(t)}{1+v(t)}\right\|_\infty^2+\frac{1}{\nu^2}\max_{[0,\tau]}\left\|\frac{\pi(t)}{1+v(t)}\right\|_\infty^2+1\right)^{d/4} \times \max\{E(\tau),k\|\mathbf{v}(0)\|_\infty\}. \tag{56}$$

Notice, however, that now a factor of the form $1/\nu^2$ appears before the maximum norm of $\Theta/(1+v)$, and this is squared, while before we had only $1/\nu$ and the power of $\theta/(1+v)$ was one. This may be important when the viscosity is low.

V. CONCLUSIONS

While the kinetic energy of the flow in Rayleigh–Bénard convection may be easily bounded by classical inequalities, the maximum of the velocity is harder to handle. As soon as the flow becomes chaotic small islands or filaments of high velocity are observed, which makes compatible the boundedness of the square mean of the velocity (the kinetic energy) with the existence of large velocity peaks. It is proved here that, provided the pressure remains uniformly bounded, so does the velocity, and its maximum may be estimated by the maxima of the temperature deviation and pressure divided by one plus the velocity modulus. Other estimates may be made in terms of certain means of the temperature, which may be considerably smaller than the temperature itself if its distribution is irregular. The bounds depend on certain powers, depending on the dimension, of the previously mentioned magnitudes and the flow viscosity.

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